Deterministic List Decoding of Reed Solomon Codes

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May 15, 2026

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Joint work with

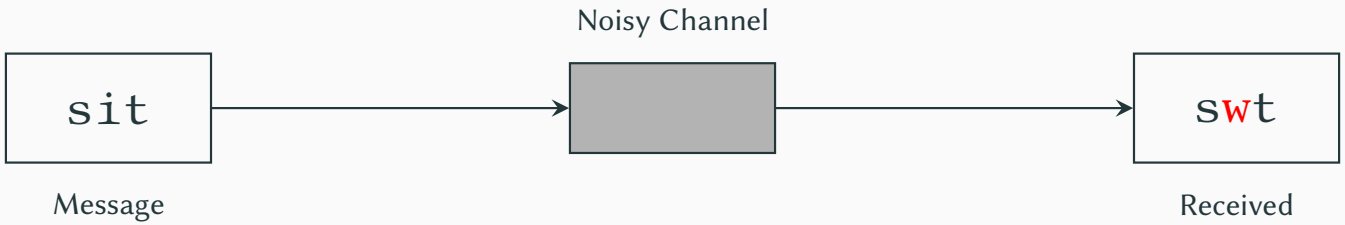
To appear in STOC 26

Introduction

What is Coding Theory?

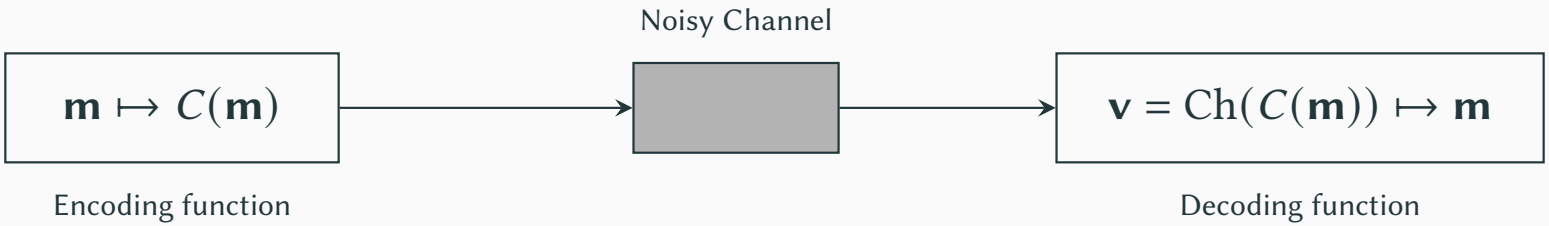
Suppose you want to send a message to your friend.

But the channel is **noisy**. Some symbols can change arbitrarily.



The symbol `i` got corrupted to `w`. What was the original message? `sit` or `sat`.

Solution: Encode m before sending; **decode v** after receiving.



A Simple Idea: Repetition Code

Idea: Send the same message 3 times: `sit` $\longrightarrow$ `sitsitsit`.

Decoding by majority

For each coordinate, take the **majority** over the three copies. A single symbol-flip anywhere is corrected.

Problem: The codeword is $3\times$ as long as the message. To handle more errors we'd need even more copies. This is **too wasteful**!

Solution

We want a **smarter encoding** that corrects errors without a large blowup in length.

With this in place, let us set up some notation…

Key Assumption

The number of errors is **bounded** — the channel cannot corrupt the entire word.

Introduction to Coding Theory

We now introduce a few terminologies used in coding theory.

Let $\mathcal{M}$ be the **message space** and Σ a finite **alphabet**. A code is a family of maps, one for each length of codeword:

- **Encoding:** $\mathrm{Enc}_n : [|\mathcal{M}|] \to \Sigma^n$
- **Decoding:** $\mathrm{Dec}_n : \Sigma^n \to [|\mathcal{M}|]$

The image $C = \mathrm{Enc}_n([|\mathcal{M}|]) \subseteq \Sigma^n$ is the **code**; its elements are **codewords**.

- **Blocklength:** n
- **Dimension:** $k = \log |C|$
- **Rate:** $R(C) = \dfrac{k}{n \log |\Sigma|}$

Rate is something like how much information of the message you are sending per bit.

Introduction to Coding Theory

The **Hamming distance** between $u, v \in \Sigma^n$ is the number of coordinates where they differ: $\Delta(u, v) = \left|\{ i : u_i \neq v_i \}\right|$

coordinate	1	2	3	4	5	6
u	0	1	1	0	1	0
v	1	1	0	0	1	1
	$\neq$	$=$	$\neq$	$=$	$=$	$\neq$

$\Rightarrow \Delta(u, v) = 3$

- **Distance of code:** $\Delta(C) = \min_{c_1 \neq c_2 \in C} \Delta(c_1, c_2)$
- **Relative distance:** $\delta(C) = \dfrac{\Delta(C)}{n}$

Intuitively: a large distance means codewords are far apart, making it easier to recover from errors.

The set $\{u \in \Sigma^n : \Delta(u, v) \leq r\}$ is a **Hamming ball** of **radius** r around v. We often speak of decoding *radius* rather than distance.

Introduction to Coding Theory

Goal: Construct codes such that

- Codewords to be "far apart" from each other $\implies$ **High Distance**.
- Redundancy to be low $\implies$ **High Rate**.

Relation between Rate and Distance (Singleton Bound)

For any code C

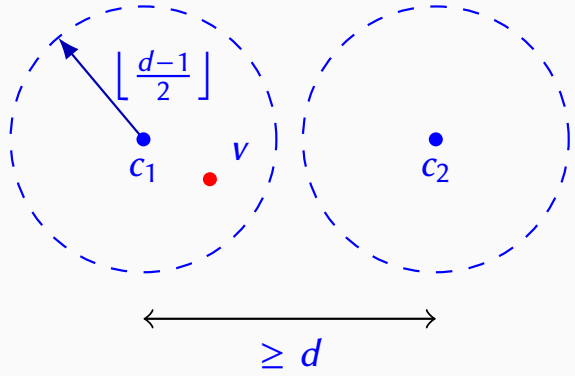
$$k \leq n - d + 1$$

Asymptotically $R + \delta \leq 1$ as n becomes very large.

Codes achieving this bound are called **Maximum Distance Separable (MDS)** codes.

Unique Decoding

Let $\Delta(C) = d$. For any $v \in \Sigma^n$ there is at most one codeword $c \in C$ such that $\Delta(v, c) \leq (d-1)/2$.



Unique Decoding Problem: Given a received word $v \in \Sigma^n$, find the unique codeword $c \in C$ such that $\Delta(v, c) < d/2$ if it exists.

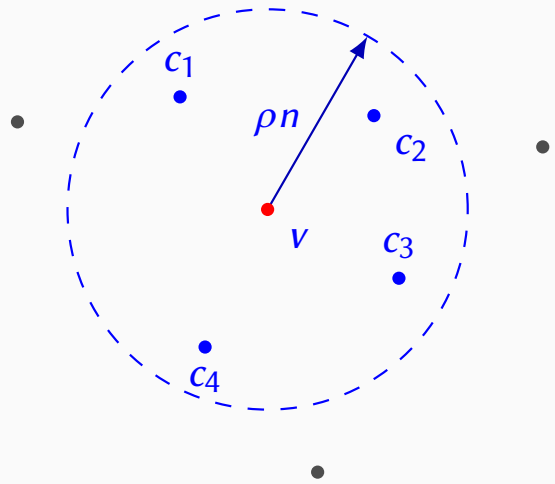
- If we go more than $d/2$ distance, multiple codewords may lie in the radius of hamming ball.

List Decoding

Definition ((ρ, L)-List Decodable)

C is (ρ, L)-list decodable if for every $v \in \Sigma^n$,

$$|\{c \in C \mid \Delta(c, v) \leq \rho n\}| \leq L$$



We write $L(v, \rho) = \{c \in C \mid \Delta(c, v) \leq \rho n\}$.

Unlike unique decoding, **multiple codewords** may lie inside the ball.

We sometimes say "m is in the list" to mean $\mathrm{Enc}_n(f) \in L(v, \rho)$, depending on the context.

List Decoding

List Decoding Problem: Given a received word $v \in \Sigma^n$, compute $L(v, \rho)$.

- **Combinatorial:** Prove $|L(v, \rho)| = \text{poly}(n)$.
- **Algorithmic:** Find all codewords in $L(v, \rho)$ in $\text{poly}(n)$ time.

Theorem (Johnson Bound)

For a code C with rate R, the list size is polynomial in n for $\rho \leq 1 - \sqrt{R}$ (also called Johnson radius).

Convention: We use **agreement** $t = 1 - \rho$ in place of error fraction ρ.

Why Polynomial-Based Codes?

We want codes with **high rate** and **large distance**. A natural question: what mathematical structure should we build on?

Answer: Polynomials are nice to use.

- The algebra of polynomials is classical and rich.
- Polynomial arithmetic can all be done very efficiently. All the basic operations in near linear time.
- Connections to algebra, number theory, and algebraic geometry give powerful tools for both encoding and decoding.

Examples: Reed-Solomon Codes, Reed-Müller Codes, BCH Codes, Multiplicity Codes…

Reed Solomon Codes (RS)

Construction

Fix a finite field $\mathbb{F}_q$ and distinct evaluation points $S = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{F}_q$.

$RS[n, k]_q$: each message m is identified with a polynomial $f_m \in \mathbb{F}_q[X]$ of degree $< k$,

and encoded by evaluating at all points of S:

$$\mathrm{Enc}(m) = \big(f_m(\alpha_1), f_m(\alpha_2), \ldots, f_m(\alpha_n)\big)$$

Rate: k/n.

Codewords form a **vector space**, so $\Delta(C)$ = min nonzero positions of any nonzero codeword = min nonzero evaluations of a nonzero $\deg < k$ polynomial.

Distance: $d = n - k + 1$

This meets the Singleton bound $d \leq n - k + 1$, making RS codes **MDS**.

Reed Solomon Decoding History

Goal: Decode in $\mathrm{poly}(n, \log q)$ time or $\mathrm{poly}(n)$ field operations.

Unique Decoding Radius: $> (n + k - 1)/2$

Johnson Radius: agreement $> \sqrt{n(k-1)}$.

Good news: all three settings above admit $\mathrm{poly}(n)$ algorithms.

Algorithm	**Agreement**	**Type**
Berlekamp–Welch	$> (n + k - 1)/2$	deterministic
Sudan '97	$> \sqrt{2n(k-1)}$	randomized
Guruswami–Sudan '99	$> \sqrt{n(k-1)}$	randomized

The **deterministic** variant of GS runs in $\mathrm{poly}(n, |\mathbb{F}|)$.

Polynomial dependence on the field characteristic ‼.

Framework of Sudan and Guruswami-Sudan

Let $w = \{(\alpha_j, \beta_j)\}_{j \in [n]}$ be the received word. Both algorithms share the same two-step structure:

Step 1: Interpolation. Find a nonzero bivariate polynomial $Q(X, Y) \in \mathbb{F}_q[X, Y]$ of $(1, k-1)$ degree D vanishing at every received point (α_j, β_j) with some multiplicity, chosen.

So that every close codeword f satisfies $Y - f(X) \mid Q(X, Y)$.

(a, b) degree of $X^i Y^j$ is $ai + bj$. So take max for polynomials.

Sudan uses multiplicity 1*; Guruswami–Sudan uses higher multiplicity* $m > 1$ *to push the radius further.*

Step 2: Factorization. Completely factor $Q(X, Y)$ over $\mathbb{F}_q$ and collect all factors of the form $Y - f(X)$

Factorization Barrier

The **interpolation step is already deterministic**. The bottleneck is the **factorization step**.

Both Sudan and Guruswami–Sudan algorithms rely heavily on the **factorization of bivariate polynomials** over finite fields.

In fact, the difficulty already appears in the **univariate** case: factoring a degree-d polynomial over $\mathbb{F}_q$ deterministically in $\mathrm{poly}(d, \log q)$ time is **open**.

- **Berlekamp**, **Cantor–Zassenhaus**: randomized $\mathrm{poly}(d, \log q)$ algorithms for univariate factoring over $\mathbb{F}_q$.
- Deterministic variants run in time **polynomial in the field characteristic**. Too slow when the characteristic is large (super polynomial)

State of Art: No truly deterministic $\mathrm{poly}(n, \log q)$ factorization or RS-list decoding algorithm is known.

Our Result

We give the **first truly deterministic** polynomial-time list decoding algorithm for Reed–Solomon codes, matching the running time of the Sudan and Guruswami–Sudan algorithms.

Key contribution: A deterministic algorithm for the factorization step: finding all factors of $Q(X, Y)$ of the form $Y - f(X)$ in $\mathsf{poly}(n)$ field operations, with no dependence on the field characteristic.

Theorem ([CHK25])

There is a deterministic algorithm that, for every finite field $\mathbb{F}$ and parameters $n > k \in \mathbb{N}$, runs in time $\mathsf{poly}(n, \log |\mathbb{F}|)$ and list decodes $RS[n, k]_{\mathbb{F}}$ from agreement greater than $\sqrt{(k-1)n}$.

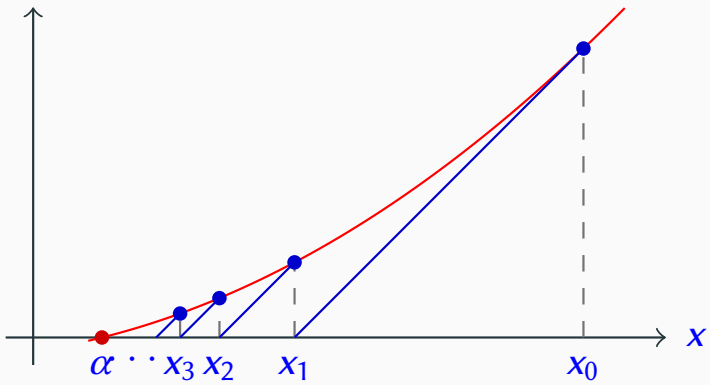
Derandomization of Sudan

Newton Iteration

Starting from x_0, each step draws a line with slope $f'(0)$ at $(x_t, f(x_t))$ and sets x_{t+1} to its x-intercept.

Each step:

1. $f'(0)$ should be non-zero
2. Evaluate $f(x_t)$ and $f'(0)$
3. $x_{t+1} = x_t - \dfrac{f(x_t)}{f'(0)}$



Converges linearly to the root α.

The same idea works for bivariate polynomials over $\mathbb{F}_q$, replacing reals with formal power series. But we take modulo an ideal.

Sudan Derandomization

Let $w = (\alpha_i, \beta_i)_{i \in [n]}$ is received word. We interpolate a polynomial $Q(X, Y)$ of $(1, k-1)$ degree $\sqrt{2n(k-1)}$ vanishes at every (α_i, β_i).

Let f is in the list. And let $j \in [n]$ such that $f(\alpha_j) = \beta_j$. Suppose Q is the polynomial from interpolation step.

If $\partial_Y Q(\alpha_j, f(\alpha_j)) \neq 0$: Then Newton Iteration from $Y_0 = f(\alpha_j)$ till Y_k gives f.

Else $\partial_Y Q(\alpha_j, f(\alpha_j)) = 0$ for all j in agreement.

Observe: $\partial_Y(Q(X, f(X)))$ has more than $\sqrt{2n(k-1)}$ roots but degree at most $\sqrt{2n(k-1)}$. Implying $\partial_Y(Q(X, f(X))) \equiv 0$

So recurse on $\partial_Y(Q(X, f(X)))$.

✓ Done.

Derandomization of Guruswami-Sudan

Local Splitting

Let $P(X, Y) \in \mathbb{F}_q[X, Y]$ with no-pure X-factors. Let $(\alpha, \beta) \in \mathbb{F}_q^2$ any point.

Suppose we are given the factorization $P = \prod_{i=1}^{s} P_i$ into irreducibles (with multiplicity)

Eg: $P(X, Y) = (Y^2 + X)(Y^2 + X + 1)^2$ then

$$P_1 = Y^2 + X, \quad P_2 = Y^2 + X + 1, \quad P_3 = Y^2 + X + 1$$

We partition $[s]$ into four sets which defines four types of factors at (α, β):

$$A(\alpha, \beta), \quad B(\alpha, \beta), \quad C(\alpha, \beta), \quad D(\alpha, \beta)$$

Let's call them A, B, C, D.

Local Splitting

Let $P = (Y^2+X+1)(Y^2+X)(Y^2+X^2+Y)(XY+1)$, $(\alpha, \beta) = (0, 0)$.

Set	Condition on $P_i(\alpha, Y)$	Example
A	$P_i(\alpha, \beta) \neq 0,\ \deg_Y P_i(\alpha, Y) \geq 1$	$Y^2 + X + 1$
B	$P_i(\alpha, Y) = \gamma(Y - \beta)^m,\ m \geq 1,\ \gamma \neq 0$	$Y^2 + X$
C	$P_i(\alpha, Y) = (Y-\beta)^m \hat{P}_i(Y),\ \hat{P}_i(\beta) \neq 0,\ \deg \hat{P}_i \geq 1$	$Y^2 + X^2 + Y$
D	$P_i(\alpha, \beta) \neq 0,\ \deg_Y P_i(\alpha, Y) = 0$	$XY + 1$

Define $P_A = \prod_{i \in A} P_i$ and similarly P_B, P_C, P_D.

Observe: If P is monic in Y then D is empty.

A Nice Observation

We interpolate a polynomial $Q(X, Y)$ of $(1, k-1)$ degree $m\sqrt{2n(k-1)}$ vanishes at every (α_i, β_i) with multiplicity $m \approx \sqrt{n(k-1)}$.

For any $j \in [n]$:

If $f(\alpha_j) = \beta_j$: Then $Y - f(X) \mid P_B$.

If $f(\alpha_j) \neq \beta_j$: Then $Y - f(X) \mid P_A$.

Remark

We will assume Q is monic in Y and has no pure X-factors.

Derandomization

If only we had: An efficient $\mathsf{poly}(n, \log q)$ time algorithm $\textsc{Split}$ to find P_A, P_B from $P, (\alpha, \beta)$ then:

Key insight: We need not fully factor Q.

If we knew **which points** have $f(\alpha_j) = \beta_j$, we can directly **interpolate** f. No factorization needed ‼

Algorithm:

1. $S \longleftarrow \{Q\}$
2. Choose $j \in [n]$ and for all $g \in S$ compute $(g_A, g_B) = \textsc{Split}(g, (\alpha_j, \beta_j))$
3. Remove g from S and put g_A, g_B in S.
4. Continue from step 2 till S stabilizes.
5. Do some interpolations to recover list.

Recover List from Stable Set

Observe: If f is in the list there is one factor $g \in S$, $Y - f(X) \mid g$.

Lemma

For all $j \in [n]$,

$$g(\alpha_j, \beta_j) = 0 \iff f(\alpha_j) = \beta_j$$

So go over all $g \in S$, find $j \in [n]$ such that $g(\alpha_j, \beta_j) = 0$, do interpolation to find appropriate f.

Problem: Stabilization can take **exponentially long**.

Notice: Every Split produces two factors, and any two elements of S share no common factor at any stage.

$\implies$ Each step **partitions** the irreducible factors of Q.

✓ Done.

Splitting Algorithm

Hensel Lifting

Given $P \equiv gh \bmod (X-\alpha)^m$ with g, h coprime mod $(X-\alpha)$, lift the factorization to twice the precision.

Theorem (Hensel)

There exist unique g', h' with $g' \equiv g$, $h' \equiv h \bmod (X-\alpha)^m$ and

$$P \equiv g'h' \bmod (X-\alpha)^{2m}$$

computable in $\mathsf{poly}(\deg P, m)$ field operations.

Iterating $\lceil \log m \rceil$ times gives a full factorization mod $(X-\alpha)^m$.

Non-monic version: [Sinhababu–Thierauf, 2021]. We extend with degree bounds for iterated lifting.

Algorithm: Lifting

$P(X, Y) \in \mathbb{F}_q[X, Y]$, monic in Y with no pure X-factors and point (α, β).

We factorize:

$$P(\alpha, Y) = \underbrace{(Y - \beta)^m}_{g_0} \cdot \underbrace{\hat{P}(Y)}_{h_0}, \quad \hat{P}(\beta) \neq 0$$

If $(Y - \beta)^m = P(\alpha, Y)$ then $P = P_A$

If $\hat{P}(Y) = P(\alpha, Y)$ then $P = P_B$

Otherwise:

Use Hensel Lifting $t = 2\log(\deg_Y P)$ times to get g_t, h_t such that

$$P \equiv g_t h_t \bmod (X - \alpha)^{2^t}, \quad g_t \equiv g_0 \bmod (X - \alpha), \quad h_t \equiv h_0 \bmod (X - \alpha)$$

Algorithm: Recursive Step

g_t, h_t may not be actual factors of P as we are viewing modulo $(X-\alpha)^{2^t}$.

Observe: Actual factor, g', h' such that $g_t = g' \cdot h''$, $h' = h'' \cdot h_t$.

E.g. $P = (Y^2 + X - 1)(Y - 1)$ at $\alpha = 0$.

Mod X: $P \equiv (Y-1)^2(Y+1)$, so Hensel gives $g_t = (Y-1)^2$, $h_t = (Y+1)$.

The extra $(Y-1)$ in g_t comes from irreducible $Y^2 + X - 1$, so $h'' = (Y-1)$.

Need to solve linear systems of the form:

$$F \equiv E \cdot h_t \bmod (X-\alpha)^{2^t}, \qquad V \equiv U \cdot g_t \bmod (X-\alpha)^{2^t}$$

Take $\deg_Y(F, V) \leq \deg_Y(P) - 1$, $\deg_X(F, V) \leq \deg_X(P)$ to ensure factors.

Observe: Both F, V have factors of P but not exactly factor of P.

So we take gcd $P_1 = gcd(P, F)$, $P_2 = gcd(P, V)$

Recurse on $P_1, P/P_1$. Then combine them to get P_A, P_B.

Final Theorem

Lemma

If the algorithm passes initial checks and has no solution of linear systems. Then $P = P_C$.

Here we mention the full version of the theorem we proved:

Theorem

For every bivariate polynomial $P(X, Y) \in \mathbb{F}_q[X, Y]$ *and point* $(\alpha, \beta) \in \mathbb{F}_q^2$ *the above algorithm outputs* (P_1, P_2) *such that* $P_1 = P_A \cdot R_1$ *and* $P_2 = P_B \cdot R_2$ *where* $R_1 R_2 \mid P_D$.

Open Problems

- Derandomize Sudan and Guruswami-Sudan in near linear time.
- What about beyond johnson bound list decoding of Reed Solomon Codes?
  Ben-Sasson-Kopparty-Radhakrishnan (2006) showed us we have no hope for low rate regime. But high/constant rate we have hope
- For beyond Johnson Bound even finding one element of the list is open.
- List decode Univariate Multiplicity Codes deterministically upto list decoding capacity in $\mathsf{poly}(1/\epsilon)\tilde{O}(n)$ time.

Thank You