LOGCONCAVE POLYNOMIALS 2:
FPRAS MATROID BASE COUNTING REPORT

Instructor: Partha Mukhopadhyay

SOHAM CHATTERJEE, RISHAV GUPTA, SRIJAN CHAKRABORTY, ADITI MUTHKHOD

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Abstract

This is a report for the paper [ALGV19] which was covered in the second half of the course ‘Expander Graphs and Application’ instructed by Partha Mukhopadhyay in the Jan-May, 2024 semester in Chennai Mathematical Institute.

Contents

1 Introduction

2 Preliminaries

2.1 Some Linear Algebra

We will first prove the Cauchy's Interlacing Theorem which will be helpfull in Lemma 3.3

Lemma 2.1 ([HJ13, Lemma 4.2.3]). *Let $S_1, \dots, S_k \subseteq V$ are the subspace of the vector space V where $\dim V = n$ such that*

$$\sum_{i=1}^{k} \dim S_i \geq (k-1)n + 1$$

Then $\dim \left(\bigcap\limits_{i=1}^{k} S_i \right) \geq 1$

Proof: Now

$$\dim \left(\bigcap_{i=1}^{k} S_i \right) = n - \dim \left(\bigcup_{i=1}^{k} \overline{S}_i \right) \geq n - \sum_{i=1}^{k} \dim \overline{S}_i = n - \sum_{i=1}^{k} (n - \dim S_i) = n - kn + \sum_{i=1}^{k} \dim S_i \geq 1$$

Hence we are done. ■

We will prove a much generalized version of Cauchy's Interlacing Theorem, i.e. Corollary 2.3. For that first we will show the a bounding relation of eigenvalues of $A + B$ by eigenvalues of A, B.

Theorem 2.2 ([HJ13, Theorem 4.3.1]). *Let $A, B \in \mathbb{C}^{n \times n}$ are hermitian. Then*

$$\lambda_i(A+B) \leq \lambda_{i-j+1}(A) + \lambda_j(B)$$

Proof: Let $\{x_i\}_{i \in [n]}$, $\{y_i\}_{i \in [n]}$, $\{z_i\}_{i \in [n]}$ are the orthonormal eigenbasis for $A, B, A+B$ respectively where the index follows the eigenvalue ordering. Now define the three subspaces:

- $S_1 := \langle x_{i-j+1}, \dots, x_n \rangle$
- $S_2 := \langle y_j, \dots, y_n \rangle$
- $S_3 := \langle z_1, \dots, z_i \rangle$

Then

$$\dim S_1 + \dim S_2 + \dim S_3 = [n - (i-j)] + [n - (j-1)] + i = 2n + 1$$

Hence by Lemma 2.1 $\dim(S_1 \cap S_2 \cap S_3) \geq 1$. Hence there exists $v \in S_1 \cap S_2 \cap S_3$ with v being unit vector. Now first we need this claim:

Claim. *For any hermitian matrix $M \in \mathbb{C}^{n \times n}$, $\forall\, x \in V$ we have*

$$x^T M x \leq \max\{\lambda_i \mid i \in [n]\}$$

Proof of Claim : Let $\{v_i\}_{i \in [n]}$ be the orthonormal eigenbasis V. Then $x = \sum\limits_{i=1}^{n} \alpha_i v_i$ for some $\alpha_i \in \mathbb{C}$. Then

$$x^T M x = \left(\sum_{i=1}^{n} \alpha_i^* v_i^T \right) \left(\sum_{i=1}^{n} \alpha_i v_i \lambda_i \right) = \sum_{i=1}^{n} |\alpha_i|^2 \lambda_i \leq \max\{\lambda_i \mid i \in [n]\}$$

■

Hence using the claim we have

$$\lambda_i(A+B) \leq v^T(A+B)v = v^T A v + v^T B v \leq \lambda_{i-j+1}(A) + \lambda_j(B)$$

And we are done. ■

Using this theorem we will prove the general version of the Cauchy's Interlacing Theorem which is just a simple corollary of Theorem 2.2:

Corollary 2.3 ([HJ13, Corollary 4.3.3])**.** *Let $A, B \in \mathbb{R}^{n\times n}$ are symmetric. Suppose B has k positive eigenvalues and l negative eigenvalues then*

$$\lambda_i(A+B) \leq \lambda_{i-k}(A)\ \forall\ i \in \{k+1,\dots,n\} \quad \textit{and} \quad \lambda_{i+l}(A) \leq \lambda_i(A+B)\ \forall\ i \in [n-l]$$

Proof: B has k positive eigenvalues. Hence $\lambda_{k+1}(B) \leq 0$. Therefore for all $i \in \{k+1,\dots,n\}$ if we take $j = k+1$ in Theorem 2.2 we have

$$\lambda_i(A+B) \leq \lambda_{i-(k+1)-1}(A) + \lambda_{k+1}(B) \leq \lambda_{i-k}(A)$$

Now B has l negative eigenvalues. Hence $-B$ has l positive eigenvalues. So $\lambda_{l+1}(-B) \leq 0$. So forall $i \in [n-l]$ we take $j = l+1$ then by Theorem 2.2 we have

$$\lambda_i(A) = \lambda_i((A+B)+(-B)) \leq \lambda_{i-(l+1)-1}(A+B) + \lambda_{l+1}(B) \leq \lambda_{i-l}(A+B)$$

Hence we have proved both the order relations. ■

Now we can easiliy see Cauchy's Interlacing Theorem is just a special case of Corollary 2.3 with $k = 1$ and $l = 0$

Theorem 2.4 (Cauchy's Interlacing Theorem, [HJ13, Corollary 4.3.9])**.** *For a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$ take $B = A + vv^T$ then*

$$\lambda_i(A) \leq \lambda_i(B) \leq \lambda_{i-1}(A)\ \forall\ i \in \{2,\dots,n\}$$

Proof: Here vv^T has only 1 positive eigenvalue. Hence we have $k = 1$ and $l = 0$ in Corollary 2.3. So $\lambda_i(B) \leq \lambda_{i-1}(A)$ and $\lambda_i(A) \leq \lambda_i(B)$. ■

With this we will now prove a result Lemma 2.5 with which will prove a helping lemma Lemma 2.6 which will help us prove Lemma 3.3:

Lemma 2.5. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $P \in \mathbb{R}^{m\times n}$. If A has atmost one positive eigenvalue then PAP^T has also one positive eigenvalue.*

Proof: Given A is symmetric. Hence there exists an unitary matrix U sush that $A = UDU^T$ for some diagonal matrix by spectral decomposition. Denote the columns of U as $v_1,\dots,v_n$ and the $i-$th diagonal entry as d_i. So we can write $A = \sum\limits_{i=1}^{n} d_i v_i v_i^T$.

If A has no positive eigenvalue then A is negative semidefinite. Then so is PAP^T since $\forall\ v \in \mathbb{R}^m$,

$$v^T PAP^T v = (P^T v)A(P^T v)^T \leq 0$$

So the lemma is valid for this case.

If A has a positive eigenvalue. Let d_i is positive. then forall $j \neq i, j \in [n]$ we have $d_j \leq 0$. Then we can write

$$A = \underbrace{\sum_{j\neq i} d_j v_j v_j^T}_{B} + \underbrace{(\sqrt{d_i} v_i)}_{w}(\sqrt{d_i} v_i)^T$$

So B is negative semidefinite matrix. Therefore PBP^T is also negative semidefinite. Now

$$PAP^T = PBP^T + (\underbrace{Pw}_{u})(Pw)^T$$

By Cauchy's Interlacing Theorem we have $\lambda_2(PAP^T) \leq \lambda_1(PBP^T) \leq \lambda_1(PAP^T)$. So we have at most one eigenvalue of PAP^T is positive. ■

Lemma 2.6. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with non negative entries and at most one positive eigenvalue and $w(i) = \sum\limits_{i=1}^{n} A_{i,j}$. Then $A \preccurlyeq \frac{ww^T}{\sum_{i=1}^{n} w_i}$*

Proof: The way w is defined we have $w = A\mathbb{1}$ where $\mathbb{1}$ is the all 1's vector. Now define $W = Diag(w)$ with keeping 1's in the diagonal for the indices with 0 in w. We will assume $A \neq 0$ since otherwise the lemma is true. Since A has non negative entries we have $w(i) \geq 0$ for all $i \in [n]$. Hence $\sqrt{W}$ exists. Then take $\mathcal{A} = W^{-\frac{1}{2}} A W^{-\frac{1}{2}}$. By Lemma 2.6 $\mathcal{A}$ has at most one positive eigenvalue. Now define the vector $\sqrt{w}(i) = \sqrt{w(i)}$. Then

$$\mathcal{A}\sqrt{w} = W^{-\frac{1}{2}} A W^{-\frac{1}{2}} \sqrt{w} = W^{-\frac{1}{2}} A \mathbb{1} = W^{-\frac{1}{2}} w = \sqrt{w}$$

So $\sqrt{w}$ is the only eigenvector of $\mathcal{A}$ with positive eigenvalue, 1.

Now

$$A \preccurlyeq \frac{ww^T}{\sum_{i=1}^n w_i} \iff \mathcal{A} \preccurlyeq \frac{\sqrt{w}\sqrt{w}^T}{\|\sqrt{w}\|^2}$$

since multiplying $W^{-\frac{1}{2}}$ to both sides to both LHS and RHS we have

$$\mathcal{A} \preccurlyeq \frac{\sqrt{w}\sqrt{w}^T}{\|\sqrt{w}\|^2} \iff A \preccurlyeq \frac{ww^T}{\|\sqrt{w}\|^2} = \frac{ww^T}{\sum_{i=1}^n w(i)}$$

So $\sqrt{w}$ is the unit vector with positive eigenvalue for both $\mathcal{A}$ and $\frac{\sqrt{w}\sqrt{w}^T}{\|\sqrt{w}\|^2}$ and for all other eigenvectors of in the orthogonal space of $\langle \sqrt{w} \rangle$, A is negative semidefinite and $\frac{\sqrt{w}\sqrt{w}^T}{\|\sqrt{w}\|^2}$ has eigenvalues 0. Therefore $\mathcal{A} \preccurlyeq \frac{\sqrt{w}\sqrt{w}^T}{\|\sqrt{w}\|^2}$. ■

Now we will prove another lemma which will be helpfull in Lemma 4.2

Lemma 2.7. *Let $A \in \mathbb{R}^{n \times n}$ symmetric with at most one positive eigenvalue. Then for any positive semidefinite matrix $B \in \mathbb{R}^{n \times n}$, BA has at most one positive eigenvalue.*

Proof: We have $B \succcurlyeq 0$. By spectral decomposition $B = U^T D U = C^C$ where $C \in \mathbb{R}^{n \times n}$. Then $BA = C^T CA = C^T(CA)$. Now $C^T(CA)$ has the same eigenvalues (with multiplicity) as $(CA)C^T = CAC^T$. A has at most one positive eigenvalue. Then by Lemma 2.5 CAC^T has at most one positive eigenvalue. Hence BA has at most one positive eigenvalue. ■

2.2 Log Concave iff At most one poistive eigenvalue

Now we will prove a very important property for a logconcave d-degree homogeneous polynomial

Theorem 2.8 ([AGV18]). *A degree d homogeneous polynomial is log concave over a set E, where $E \subseteq \mathbb{R}^n_{>0}$ if and only if $(\nabla^2 p)(x)$ has at most one positive eigen value at all $x \in E$.*

Proof: First note the following expression

$$\nabla^2(\log(p)) = \frac{\nabla^2 p}{p} - \frac{(\nabla p) \cdot (\nabla p)^T}{p^2}$$

Also recall the Euler's identity for a degree d homogeneous polynomial p

$$d \cdot p(x_1, x2, \cdots, x_n) = \sum_{i=1}^n x_i \cdot \frac{\delta p}{\delta x_i}.$$

Also note that when we apply Euler's identity on the gradient function slot wise then we get that

$$(d-1) \cdot \nabla p(x_1, x2, \cdots, x_n) = \sum_{i=1}^n x_i \cdot \nabla(\frac{\delta p}{\delta x_i})$$

Let, $Q(\vec{x}) = \nabla^2 p(\vec{x})$ From the above expressions we get that

$$(d-1) \cdot \nabla p(x_1, x2, \cdots, x_n) = \sum_{i=1}^n x_i \cdot \nabla(\frac{\delta p}{\delta x_i})$$

$$\implies (d-1) \cdot \nabla p(\vec{x}) = Q(\vec{x}) \cdot \vec{x}$$

Similarly, we get that

$$d \cdot p(\vec{x}) = (\vec{x})^T \cdot \nabla p(\vec{x})$$

$$\implies d(d-1) \cdot p(\vec{x}) = (\vec{x})^T \cdot Q(\vec{x}) \cdot \vec{x}$$

So what we will do is substitute p and ∇p in terms of Q in $\nabla^2(\log(p))$ to get

$$\nabla^2(\log(p)) = d \cdot (\frac{(d-1)(x^TQx)(Q) - (d)(Q \cdot x \cdot x^T \cdot Q^T)}{(x^TQx)^2})$$

Note that the denominator is always positive so it will not play any role in our analysis. We will first prove the backward direction that if p is log concave at $\vec{x}$ then $Q(\vec{x})$ has atmost one positive eigen value .
If p is log concave at $\vec{x}$ then $\nabla^2(\log(p))$ is negative semidefinite. Now note the following that

$$Q = p \cdot \nabla^2(\log(p)) + \frac{(\nabla p)(\nabla p)^T}{p}$$

Since we know that p is a polynomial with non negative coefficients and $\vec{x} \in \mathbb{R}^n_{>0}$ implies $p(\vec{x})$ is positive which implies

$$Q = B + u \cdot u^T$$

where B is a negative semidefinite matrix which implies by Cauchy interlacing theorem that Q has atmost one positive eigenvalue.

Now we will show the forward direction that if Q has at most one positive eigen value at $\vec{x}$ then $\nabla^2(\log(p(\vec{x})))$ is negative semidefinite . Now from previous expression we only need to show that if Q has atmost one positive eigen value then $d \cdot (\frac{(d-1)(x^TQx)(Q)-(d)(Q \cdot x \cdot x^T \cdot Q^T)}{(x^TQx)^2})$ is negative semidefinite. Now the denominator is use less so it reduces to showing $N = (d-1)(x^TQx)(Q) - (d)(Q \cdot x \cdot x^T \cdot Q^T)$ is negative semidefinite . We will in fact prove a stronger result we will prove that $M = ((x^TQx)(Q) - (Q \cdot x \cdot x^T \cdot Q^T))$ is negative semidefinte since $N = (d-1)M - (Q \cdot x \cdot x^T \cdot Q^T)$ and since the set of negative semidefinte matrices is closed under addition we will be done .

So now we only need to show that if Q has atmost one positive eigenvalue then $M = ((x^TQx)(Q) - (Q \cdot x \cdot x^T \cdot Q^T))$ is negative semidefinite.
For this take a an arbitrary vector $y \in \mathbb{R}^n$, Now we need to show that $(x^TQx)(y^TQy) - (y^TQx)^2 \leq 0$.
For proving this, consider a $(n \times 2)$ matrix P with columns x and y. Then we get that

$$P^TQP = \begin{bmatrix} x^TQx & x^TQy \\ y^TQx & y^TQy \end{bmatrix}$$

So now we need to show $det(P^TQP) \leq 0$. Now if y^TQy is non positive then we are done because then $(x^TQx)(y^TQy) - (y^TQx)^2 \leq 0$ will be true as $(x^TQx) \geq 0$.(Since x is a positive vector and Q has all its entries positive).

So now we are restricted to the case where $y^TQy > 0$. Now note that for a 2×2 symmetric matrix if both of the diagonal entries are non-negative then it has a non-negative eigenvalue . (Note P^TQP is symmetric since Q is symmetric and for proving the above fact just use the forms of root of the characteristic polynomial). So in the case where $y^TQy > 0$ we have existence of a positive eigenvalue. Now if rank of P was 1 then the above determinant would be zero and we would be done so we are restricted to the case where rank of P is 2. The dimension of the vector space over which Q is negative semidefinite is at least $(n-1)$ which implies that there exists a vector in the intersection of image of P and the vector space over which Q is negative semidefinite as their dimensions sum up to $(n+1) > n$ hence we get that if $u = Pv$ is that vector then $u^TQu = v^T(P^TQP)v \leq 0$ which implies that (P^TQP) must have a non positive eigenvalue. Now we have existence of a non negative and a non positive eigen value in the case where $(y^TQy) > 0$ hence we get that $det(P^TQP) \leq 0$ in this case hence we are done. ■

2.3 Matroids

Definition 2.1 (Matroid). *A matroid $M = ([n], \mathcal{I})$ is a set (called ground set $[n]$), along with a nonempty subset $\mathcal{I}$ of its power set, with the following properties:*

- ***Downward closure**: If $S \subseteq T$, and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.*
- ***Extension property**: If $S, T \in \mathcal{I}$, and $|S| < |T|$ then $\exists i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{I}$.*

The subsets in $\mathcal{I}$ are called independent.

The *rank* of a subset $S \subset [n]$, (rank(S)), is the size of any maximal independent set of M contained in S. Thus $S \in \mathcal{I} \iff \text{rank}(S) = |S|$. If $\text{rank}(M) := \text{rank}([n]) = r$, then any set $S \in \mathcal{I}$ of size r is called a *basis* of M.

An element $i \in [n]$ is a *loop* if $\{i\} \notin \mathcal{I}$, that is, $\{i\}$ is dependent. Two non-loops $i, j \in [n]$ are *parallel* if $\{i, j\} \notin \mathcal{I}$, that is, $\{i, j\}$ is dependent.

Being parallel defines an equivalence relation on the non-loop elements of $[n]$ and say $S_1, \cdots, S_k$ are the corresponding equivalence classes. The **matroid partition property** says that for any matroid the non-loops can be partitioned into sets, such that any two non-loops are parallel iff they belong to the same set.

We define the ***contraction*** M/S for a matroid $M = ([n], \mathcal{I})$, and $S \in \mathcal{I}$ as the matroid $M/S = ([n] \setminus S, \mathcal{I}')$, where $\mathcal{I}' = \{T \subseteq [n] \setminus S | T \cup S \in \mathcal{I}\}$.

2.4 Simplicial Complexes

Let $[n]$ be the ground set. A *simplicial complex* X on the ground set is a downward closed family over $2^{[n]}$, i.e. $\sigma \in X, \tau \subseteq \sigma \implies \tau \in X$. $X = \{\sigma\}$, σ are called *faces* in Linear Programming, and *simplices* in Topology.

The *dimension* is defined as follows: $dim(\sigma) := |\sigma|$; $dim(\phi) = 0$. For an integer k,

$$X(k) := \{\sigma \in X | dim(\sigma) = k\}.$$

$dim(X) := max\{k | X(k) \neq \phi\}$. X is said to be *pure* if each maximal $\sigma \in X$ has dimension equal to $dim(X)$. This paper only considers pure simplicial complexes.

The *link* of a face τ is defined as

$$X_\tau := \{\sigma \setminus \tau | \sigma \in X, \tau \subset \sigma\}.$$

If X is pure, $dim(X) = d$, and $\tau \in X(k)$, then X_τ is a pure simplicial complex of dimension $d - k$.

For any matroid $M = ([n], \mathcal{I})$ of rank r, the independent sets $\mathcal{I}$ form a pure r-dimensional simplicial complex on $[n]$ called its *independence (or matroid) complex*. For any $S \in \mathcal{I}$, the contraction $M/S = ([n], \mathcal{I}_S)$.

We can equip a simplicial complex with a weight function: $w : X \to \mathbb{R}_{>0}$. w is said to be *balanced* if for all k, for every non maximal face $\tau \in X(k)$,

$$w(\tau) = \sum_{\sigma \in X(k+1): \tau \subset \sigma} w(\sigma).$$

For a pure simplicial complex X we can define a balanced weight function by assigning arbitrary positive weights to maximal faces and defining the weight of each lower dimensional face recursively. Indeed, if X is a pure simplicial complex of dimension d and w is a balanced weight function, then, for any $\tau \in X(k)$,

$$w(\tau) = (d-k)! \sum_{\sigma \in X(d): \tau \subset \sigma} w(\sigma).$$

Any balanced weight function on X induces a weighted graph on the vertices of X as follows. Taking inspiration by graph theory, we call $X(1)$ *vertices*, and $X(2)$ *edges*, and $G = ((X(1), X(2))$ a *1-skeleton* of X. Then, restricting w to $X(1)$ and $X(2)$ determines a weighted graph, where $w(v)$ gives the weighted degree of each $v \in X(1)$.

3 Walks on Simplicial Complexes

For a pure d-dimensional complex X, and a balanced weight function $w : X \to \mathbb{R}_{>0}$, we define two random walks called the *upper k-walk* and *lower k-walk*. These random walks are defined on G_k, a bipartite graph with the two partitions as $X(k)$ and $X(k+1)$. First, define $P_k^{\uparrow}, P_k^{\downarrow}$ as follows: $P_k^{\uparrow} \in \mathbb{R}^{X(k) \times X(k+1)}$ is a stochastic matrix for a random walk from $X(k+1)$ to $X(k)$, where from $\sigma \in X(k+1)$, we drop an element uniformly at random and go to some $\tau \in X(k)$. $P_k^{\downarrow} \in \mathbb{R}^{X(k+1)\times X(k)}$ is a stochastic matrix for a random walk from $X(k)$ to $X(k+1)$, where from $\tau \in X(k)$, we go to $\tau \subset \sigma \in X(k+1)$ with probability $w(\sigma)/w(\tau)$.

Definition 3.1 (Upper k-walk). *This is the walk defined by the concatenation of the walks in $P_k^{\downarrow}, P_k^{\uparrow}$. the stochastic matrix for this walk is defined by $P_k^{\wedge} = P_k^{\uparrow} P_k^{\downarrow}$.*

Definition 3.2 (Lower k-walk). *This is the walk defined by the concatenation of the walks in $P_k^{\uparrow}, P_k^{\downarrow}$. the stochastic matrix for this walk is defined by $P_k^{\vee} = P_k^{\downarrow} P_k^{\uparrow}$.*

Now, let us formally write down the entries of $P_k^{\wedge}$ and $P_{k+1}^{\vee}$. Given a simplex $\tau \in X(k)$, for $1 \leqslant k < d$,

$$P_k^{\wedge}\left(\tau, \tau'\right) = \begin{cases} \frac{1}{k+1}, & \text{if } \tau = \tau' \\ \frac{v(\tau \cup \tau')}{(k+1) w(\tau)}, & \text{if } \tau \cup \tau' \in X(k+1) \\ 0, & \text{otherwise} \end{cases}$$

Note that the upper walk is not defined for $k = d$, because there is no $(d+1)$-dimensional simplex in X. Analogously, given $\sigma \in X(k+1)$, for $1 \leqslant k < d$,

$$P_{k+1}^{\vee}\left(\sigma, \sigma'\right) = \begin{cases} \sum_{\tau \in X(k): \tau \tau \sigma} \frac{w(\sigma)}{(k+1) w(\tau)}, & \text{if } \sigma = \sigma' \\ \frac{w(\sigma')}{(k+1) w(\sigma \cap \sigma')}, & \text{if } \sigma \cap \sigma' \in X(k) \\ 0, & \text{otherwise} \end{cases}$$

We observe that the corresponding random walks are reversible w.r.t the weight function w, i.e., for all $\tau, \tau' \in X(k)$, we have

$$w(\tau) P_k^{\wedge}\left(\tau, \tau'\right) = w\left(\tau'\right) P_k^{\wedge}\left(\tau', \tau\right) \quad w(\tau) P_k^{\vee}\left(\tau, \tau'\right) = w\left(\tau'\right) P_k^{\vee}\left(\tau', \tau\right).$$

This implies that both chains have the same stationarity distribution where the probability of $\tau \in X(k)$ is proportional to $w(\tau)$.

Lemma 3.1. *For any $1 \leqslant k < d$, $P_k^{\wedge}$ and $P_{k+1}^{\vee}$ are stochastic, self-adjoint w.r.t. the w-induced inner product, PSD, and have the same (with multiplicity) non-zero eigenvalues.*

Proof: Let P_k be the transition probability matrix of the simple random walk on G_k. Since G_k is bipartite and we can write

$$P_k = \begin{bmatrix} 0 & P_k^{\downarrow} \\ P_k^{\uparrow} & 0 \end{bmatrix}$$

Note that P_k is self-adjoint w.r.t. the weight-induced inner product given by weights of the stationary distribution. It follows that P_k is self-adjoint w.r.t. the inner product

$$\langle \phi, \psi \rangle = \sum_{\tau \in X(k)} w(\tau) \phi(\tau) \psi(\tau) + (k+1) \sum_{\sigma \in X(k+1)} w(\sigma) \phi(\sigma) \psi(\sigma)$$

Observe that

$$P_k^2 = \begin{bmatrix} P_k^{\downarrow} P_k^{\uparrow} & 0 \\ 0 & P_k^{\uparrow} P_k^{\downarrow} \end{bmatrix}$$

$\therefore$ P_k^2 is PSD (since squares of eigenvalues are non-negative) and stochastic. Since $P_k^{\wedge}$ and $P_{k+1}^{\vee}$ correspond to two step walks on G_k, indeed we can write

$$P_k^{\wedge} = P_k^{\uparrow} P_k^{\downarrow}$$
$$P_{k+1}^{\vee} = P_k^{\downarrow} P_k^{\uparrow}$$

It follows that both matrices are self-adjoint w.r.t. the w-induced inner product, are PSD, and stochastic, and have the same eigenvalues (since AB and BA have the same nonzero eigenvalues with multiplicity). ■

Observe that $P_1^{\wedge}$ is the transition probability matrix of the simple (1/2)-lazy random walk on the weighted 1 -skeleton of X where the weight of each edge $e \in X(2)$ is $w(e)$. We also need to consider the non-lazy variant of this random walk, given by the transition matrix

$$\tilde{P}_1^{\wedge} = 2\left(P_1^{\wedge} - I/2\right)$$

Similarly, for any face $\tau \in X(k)$, we define the upper random walk on the faces of the link X_τ. Specifically, let $P_{\tau,1}^{\wedge}$ denote the transition matrix of the upper walk, as above, on the 1-dimensional faces of X_τ, and

$$\tilde{P}_{\tau,1}^{\wedge} = 2\left(P_{\tau,1}^{\wedge} - I/2\right)$$

be the transition matrix for the non-lazy version.

Definition 3.3 (Local Spectral Expanders). *For $\lambda \geqslant 0$, a pure d-dimensional weighted complex (X, w) is a λ-local-spectral-expander if for every $0 \leqslant k < d-1$, and for every $\tau \in X(k)$, we have $\lambda_2\left(\tilde{P}'_{\tau,1}\right) \leqslant \lambda$.*

Now we state the main theorem of this section. $P_k^{\wedge}$ has very few "big" eigenvalues. For example, $P_k^{\wedge}$ has exactly one eigenvalue strictly larger than $\frac{k}{k+1}$ corresponding to the maximum eigenvalue (which has value 1) and at most $n = |X(1)|$ eigenvalues strictly larger than $\frac{k-1}{k+1}$. Hence, $P_k^{\wedge}$ has at most $n-1$ eigenvalues between $\frac{k-1}{k+1}$ and $\frac{k}{k+1}$. Note that the significance of this theorem is that we can establish an estimate on all eigenvalues of $P_k^{\wedge}$. Formally:

Theorem 3.2. *Let (X, w) be a pure d-dimensional weighted 0 -local spectral expander and let $0 \leqslant k < d$. Then, for all $-1 \leqslant i \leqslant k, P_k^{\wedge}$ has at most $|X(i)| \leqslant \binom{n}{i}$ eigenvalues of value $> 1 - \frac{i+1}{k+1}$, where for convenience, we set $X(-1) = \varnothing$ and $\binom{n}{-1} = 0$. In particular, the second largest eigenvalue of $P_k^{\wedge}$ is at most $\frac{k}{k+1}$.*

For the proof, we will need the following lemma. We recall that the inner product on the space $\mathbb{R}^{X(k)}$ is given by $\langle \phi, \psi \rangle = \sum_{\tau \in X(k)} w(\tau)\phi(\tau)\psi(\tau)$, and that being self-adjoint, PSD, and the Loewner order are defined w.r.t. this inner product.

Lemma 3.3. $P_k^{\wedge} \preccurlyeq \frac{k}{k+1} P_k^{\vee} + \frac{1}{k+1} I$ *for all* $0 \leqslant k < d$.

Proof: For convenience, let $M = P_k^{\wedge} - \left(\frac{k}{k+1} P_k^{\vee} + \frac{1}{k+1} I\right)$. Fix $\eta \in X(k-1)$. We will first consider submatrices M_η whose entries are given by the following:

$$M_\eta(\tau, \sigma) = \begin{cases} M(\tau, \sigma), & \text{if } \tau \neq \sigma, \eta = \tau \cap \sigma \\ -\frac{1}{k+1} \cdot \frac{w(\tau)}{w(\eta)}, & \text{if } \tau = \sigma, \tau \supset \eta \\ 0, & \text{otherwise} \end{cases}$$

Note that $M = \sum_{\eta \in X(k-1)} M_\eta$ and hence, it suffices to prove that $M_\eta \preccurlyeq 0$ for every $\eta \in X(k-1)$. Fix $\eta \in X(k-1)$. Let $\tau, \sigma \in X(k)$ with $\tau \neq \sigma$ and $\tau \cap \sigma = \eta$. Then

$$M_{\tau\cap\sigma}(\tau, \sigma) = M(\tau, \sigma) = \frac{1}{k+1}\left(\frac{w(\tau \cup \sigma)}{w(\tau)} - \frac{w(\sigma)}{w(\tau \cap \sigma)}\right) = \frac{1}{k+1}\left(\frac{w(\tau \cup \sigma) w(\tau \cap \sigma) - w(\tau) w(\sigma)}{w(\tau) w(\tau \cap \sigma)}\right)$$

Furthermore, by definition, if $\tau \in X(k)$ with $\tau \supset \eta$, then $M_\eta(\tau, \tau) = -\frac{1}{k+1} \cdot \frac{w(\tau)}{w(\eta)}$. A matrix calculation reveals that

$$M_\eta = \frac{1}{(k+1) w(\eta)} \operatorname{diag}\left(w_\eta\right)^{-1} \cdot \left(w(\eta) \cdot A_\eta - w_\eta w_\eta^{\top}\right)$$

where w_η is the $|X(k)|$-dimensional vector whose non-zero entries are $w(\tau)$ for $\tau \supset \eta$, and A_η is the $|X(k)| \times |X(k)|$ matrix whose non-zero entries are $w(\tau \cup \sigma)$ for $\tau, \sigma \in X(k)$ satisfying $\tau \cup \sigma \in X(k+1)$ and $\tau \cap \sigma = \eta$. Note that M_η is NSD w.r.t. the inner product defined by w, if and only if $\operatorname{diag}\left(w_\eta\right) M_\eta$ is NSD in the usual sense, because for any v

$$\langle v, M_\eta v \rangle = v^{\top} \operatorname{diag}\left(w_k\right) M_\eta v = v^{\top} \operatorname{diag}\left(w_\eta\right) M_\eta v,$$

where w_k is the vector of w values on $X(k)$ and for the last equality we used that w_k is the same as w_η on all $\tau \supset \eta$.

Thus, it suffices to prove that $A_\eta \preccurlyeq \frac{w_\eta w_\eta^\top}{w(\eta)}$. We view A_η as the weighted adjacency matrix of the 1 -skeleton (which we recall is a graph) of the link X_η. Then $\tilde{P}^\wedge_{\eta,1} = \frac{1}{k+1} \operatorname{diag}\left(w_\eta\right)^{-1} A_\eta$ gives its non-lazy simple random walk matrix. As (X, w) is a 0 -local spectral expander, $\tilde{P}^\wedge_{\eta,1}$ has at most one positive eigenvalue, whence $A_\eta = (k+1) \operatorname{diag}\left(w_\eta\right) \cdot \tilde{P}^\wedge_{\eta,1}$ has at most one positive eigenvalue.

Finally, observe that the weights being balanced enforces that $w(\tau) = \sum_{\sigma \in X(k): \tau \cup \sigma \in X(k+1)} w(\tau \cup \sigma)$ and $w(\eta) = \sum_{\tau \in X(k): \tau \supset \eta} w(\tau)$. That $A_\eta \preccurlyeq \frac{w_\eta w_\eta^\top}{w(\eta)}$ then follows immediately from Lemma 2.6. ■

Proof: [Proof of the Theorem] We go by induction on k. The case $k = 0$ is trivial, as $P^\wedge_0$ is 1×1. When $k = 1$, we have $P^\wedge_1 = \frac{1}{2}\left(\tilde{P}^\wedge_1 + I\right)$. As (X, w) is a 0 -local spectral expander, $\tilde{P}^\wedge_1$ has exactly one positive eigenvalue, with value 1 . Hence, $P^\wedge_1$ has eigenvalue 1 with multiplicity 1 . All other eigenvalues of $P^\wedge_1$ are less than or equal to $1/2$, of which, there are $|X(1)| - 1$ many. Thus, the base case holds.

Assume the claim holds for some $d - 1 > k \geqslant 0$. Recall by Lemma 3.1, $P^\vee_{k+1}$ has the same non-zero eigenvalues as $P^\wedge_k$. By Lemma 3.5,

$$P^\wedge_{k+1} \preccurlyeq \frac{k+1}{k+2} P^\vee_{k+1} + \frac{1}{k+2} I$$

For $-1 \leqslant i \leqslant k, P^\vee_k$ as at most $|X(i)|$ eigenvalues $> 1 - \frac{i+1}{k+1}$. Hence, $P^\wedge_{k+1}$ has at most $|X(i)|$ eigenvalues $> \frac{k+1}{k+2} \cdot \left(1 - \frac{i+1}{k+1}\right) + \frac{1}{k+2} = 1 - \frac{i+1}{k+2}$. For $i = k+1$, we trivially have that $P^\wedge_{k+1}$ has at most $|X(k+1)|$ eigenvalues > 0, as $P^\wedge_{k+1}$ is $|X(k+1)| \times |X(k+1)|$. ■

4 From Strongly Log-Concave Polynomials to Local Spectral Expander

4.1 Logconcave polynomial to 0-local Expander

Suppose we have a $d-$homogeneous strongly log concave polynomial $p(x_1, \ldots, x_n) = \sum\limits_{S \subseteq [n], |S| = d} c_S x^S \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ with $c_S \geq 0$. We can construct a pure $d-$dimensional simplicial complex (X^p, w) with a weight assignment w where the $X^p(d) = \{S \subseteq [n] \colon |S| = d, c_S \neq 0\}$ then taking all possible subsets of elements in $X^p(d)$. Here for any $S \in X^p(d)$, the weight assignment is $w(S) = c_S$ then for any subset of the elements in $X^p(d)$ the weight assignment is defined following the weight scheme.

Now define $\forall\ \tau \in X^p$ denote $p_\tau = \left(\prod\limits_{i \in \tau} \partial_i\right) p$

Lemma 4.1. *For all* $0 \leq k \leq d, \forall\ \tau \in X^p(k)$ *we have* $w(\tau) = (d-k)! p_\tau(\mathbb{1})$

Proof: We will prove this using induction on $d - k$. For base case, $K = d$. Then $p_\tau = c_\tau$ then $w(\tau) = 0! c_\tau = c_\tau$ For inductive step let $\tau \in X^p(k)$ then

$$\begin{aligned}
w(\tau) &= \sum_{\sigma \in X^p(k+1)} w(\sigma) = \sum_{i \in X^p(1)} w(\tau + i) \\
&= \sum_{i \in X^p(1)} (d - (k+1))! p_{\tau+i}(\mathbb{1}) \\
&= (d - (k+1))! \sum_{i \in X^p(1)} \partial_i p_\tau(\mathbb{1}) \\
&= (d - (k+1))! \sum_{i \in X^p(1)} (x_i \partial_i p_\tau)(\mathbb{1}) \\
&= (d - (k+1))! (d-k) p_\tau(\mathbb{1}) && \text{[Euler's Identity]} \\
&= (d-k)! p_\tau(\mathbb{1})
\end{aligned}$$

■

Now we will show that the simplicial complex generated by the strongly logconcave polynomial is a good expander.

Proposition 4.2. *$p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be multilinear $d-$homogeneous strongly logconcave polynomial then the simplicial complex generated by p, (X^p, w) is a $0-$local expander.*

Proof: p is strongly logconcave. Hence by Theorem 2.8 we have $\nabla^2 p_\tau(\mathbb{1})$ has at most one positive eigenvalue. Now denote

$$\tilde{\nabla}^2 p_\tau = \frac{1}{d-k-1} Diag(\nabla p_\tau(\mathbb{1})^{-1} \nabla^2 p_\tau(\mathbb{1})$$

Now we claim:

Claim. $\tilde{\nabla}^2 p_\tau = \tilde{P}^{\wedge}_{\tau,1}$

Proof: We have that $\tilde{P}^{\wedge}_{\tau,1}(i,j) = \frac{w_\tau(i,j)}{w_\tau(i)} = \frac{w(\tau+i+j)}{w(\tau+i)}$. Then using Lemma 4.1 we have for

$$\tilde{\nabla}^2 p_\tau(i,j) = \frac{\partial_i \partial_j p_\tau(\mathbb{1})}{(d-k-1)(\partial_i p_\tau)(\mathbb{1})} = \frac{\dfrac{w(\tau+i+j)}{(d-k-2)!}}{(d-k-1)\dfrac{w(\tau+i)}{(d-k-1)!}} = \frac{w(\tau+i+j)}{w(\tau+i)}$$

Hence we have $\tilde{P}^{\wedge}_{\tau,1} = \tilde{\nabla}^2 p_\tau$. ■

Now $\nabla p + \tau(\mathbb{1})$ has non-negative entries. Hence we have $Diag(\nabla p_\tau(\mathbb{1})) \succcurlyeq 0$ and therefore $Diag(\nabla p_\tau(\mathbb{1}))^{-1} \succcurlyeq 0$. We also have $\nabla^2 p_\tau(\mathbb{1})$ has at most one positive eigenvalue. Therefore by Lemma 2.7 we have $Diag(\nabla p_\tau(\mathbb{1}))^{-1} \nabla^2 p_\tau(\mathbb{1})$ has at most one eigenvalue. Hence $\tilde{\nabla}^2 p_\tau = \tilde{P}^{\wedge}_{\tau,1}$ has at most one positive eigenvalue and therefore $\lambda_2(\tilde{P}^{\wedge}_{\tau,1}) \leq 0$. Now since τ here arbitrary we have X^p is a 0-local expander. ■

Hence if we have a $d-$homogeneous multilinear strongly logconcave polynomial then we have a $0-$local expander.

4.2 Distribution to Markov Chain on Bases with Good Spectral Gap

Suppose we have a probability distribution $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ on subsets of $[n]$. Then from μ. we can construct a multilinear polynomial

$$g_\mu(x_1, \ldots, x_n) := \sum_{S \subseteq [n]} \mu(S) x^S$$

Here g_μ is called the generating polynomial of μ. Then we call μ is $d-$homogeneous if g_μ is $d-$homogeneous i.e. $\mu(S) > 0 \implies |S| \leq d$. Also we call μ to be strongly logconcave if g_μ is strongly logconcave

So suppose we have a $d-$homogeneous probability distribution. Then we have a natural Monte Carlo Markov Chain $\mathcal{M}_\mu$ on the support of μ where

- State space of $\mathcal{M}_\mu = Supp(\mu)$
- For $\tau \in Supp(\mu)$ we first drom on elemetn $i \in \tau$ choosen uniformly at random from τ then among all $\sigma \supseteq \tau - i$ in support of μ we choose σ with the probability proportional to $\mu(\sigma)$ i.e we do a $P^{\vee})d$ at the top layer of the generated simplicial complex of g_μ.
- $\mathcal{M}_\mu$ is reversible.

Definition 4.1 (Total Variation Mixing Time)**.** *For a markov chain $\mathcal{M}$ with state τ and $\epsilon > 0$ total variation mixing time of $\mathcal{M}$ started at τ with transition probability matrix P and stationary distribution π*

$$t_\tau(\epsilon) := \min\{t \in \mathbb{Z}_{\geq 0} \mid \|P^t(\tau, \cdot) - \pi\| < \epsilon$$

where $P^t(\tau, \cdot)$ is the distribution of chain started at τ at time t.

Then we have a bound on total variation of mixing time with the following theorem:

Theorem 4.3 ([DS91])**.** *For a reversible markov chain $\mathcal{M} = (\Omega, P, \pi)$, $\epsilon > 0$ and any starting state $\tau \in \Omega$ we have*

$$t_\tau(\epsilon) \leq \frac{1}{1 - \lambda^*(P)} \log\left(\frac{1}{\epsilon \pi(\tau)}\right)$$

where $\lambda^* = \max\{|\lambda_2|, |\lambda_n|\}$

Now we will show one of the main theorem that if we have a d-homogeneous strongly logconcave probability distribution μ then the markov chain $\mathcal{M}_\mu$ has good spectral gap

Theorem 4.4. *Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a $d-$homogeneous strongly logconcave probability distribution. If P_μ denotes the transition probability matrix of $\mathcal{M}_\mu$ and X_μ be the generated simplicial complex. The for every $0 \leq k \leq d$, P_μ has atmost $|X_\mu(k)| \leq \binom{n}{k}$ eigenvalues of value $> 1 - \frac{k+1}{d}$.*

In particular $\mathcal{M}_\mu$ has spectral gap at least $\frac{1}{d}$ and if $\tau \in Supp(\mu)$ and $0 < \epsilon < 1$ then

$$t_\tau(\epsilon) \leq d \log\left(\frac{1}{\epsilon \pi(\tau)}\right)$$

Proof: μ is $d-$homogeneous strongly logconcave. Hence G_μ is $d-$homogeneous strongly logconcave polynomial. Then by Proposition 4.2, X_{mu} is a 0-local expander.

Now we have $P_\mu = P_d^\vee$. $P_d^\vee$ and $P_{d-1}^\wedge$ have same eigenvalues with multiplicities by Lemma 3.1. Now by Theorem 3.2 $P_{d-1}^\wedge$ has at most $|X_\mu(k)| \leq \binom{n}{k}$ eigenvalues of value $> 1 - \frac{k+1}{(d-1)+1} = 1 - \frac{k+1}{d}$. Hence if we take $k = 0$ then at most 1 eigenvalue has value $> 1 - \frac{1}{d}$. Therefore the rest of the eigenvalues are $\leq 1 - \frac{1}{d}$. Hence the spectral gap of $\mathcal{M}_\mu$ is at least $\frac{1}{d}$ and

$$\lambda^*(P_d^\vee) \leq 1 - \frac{1}{d} \iff \frac{1}{1 - \lambda^*(P_d^\vee)} \leq d \implies t_\tau(\epsilon) \leq \frac{1}{1 - \lambda^*(P_d^\vee)} \log\left(\frac{1}{\epsilon\pi(\tau)}\right) \leq d \log\left(\frac{1}{\epsilon\pi(\tau)}\right)$$

■

blah blah blah testing

5 Proof of theorem 2.16

Theorem 5.1. *Take $p \in \mathbb{R}[x_1, \cdots x_n]$ be a $d-$homogeneous polynomial such that:*

1. *for any $0 \leq k \leq d-2$ and any $(i_1, \cdots, i_k \in [n]^k, \partial_{i_1} \cdots \partial_{i_k} p)$ is indecomposable.*
2. *for any $(i_1, \cdots i_{d-2}) \in [n]^{d-2}$, the quadratic $\partial_{i_1} \cdots \partial_{i_{d-2}} p$ is either identically zero or log-concave at $\mathbb{1}$.*

Then p is strongly log-concave at $\mathbb{1}$.

We use the following theorems and propositions to prove this result.

Theorem 5.2 (Cheeger's inequality). *For any d-regular weighted graph $G = (V, E, w)$*

$$\frac{d - \lambda_2(A_G)}{2} \leq cond(G) \leq \sqrt{2(d - \lambda_2(A_G))}$$

where A_G is the weighted adjacency matrix of G given by $(A_G)_{ij} = w(\{i, j\})$.

Corollary 5.1. *$A \in \mathbb{R}^{n \times n}$ is a stochastic matrix corresponding to a reversible Markov chain with the property that $\sum_{|\{i,j\} \cap S| = 1} A_{i,j} > 0$ for all subsets $\phi \subseteq S \subseteq [n]$, then $\lambda_2(A) < 1$.*

Proposition 5.1. *A degree-d homogeneous polynomial $p \in \mathbb{R}[x_1, \cdots x_n]$ with nonnegative coefficients is log-concave over $\mathbb{R}^n_{>0} \iff (\nabla^2 p)(x)$ has at most one positive eigenvalue at all $x \in \mathbb{R}^n_{>0}$.*

Proof: The proof proceeds by induction on the degree of p. For the case $deg(p) < 2$ the theorem clearly holds, hence it serves as the base case. So for $deg(p) =: d \geq 3$ we prove that if the theorem holds for d then the theorem also holds for the case $d + 1$. For notational convenience we denote $p_i = \partial_i p$. So we assume as the induction hypothesis that for all i, p_i is strongly log-concave at $\mathbb{1}$.

Now note that $\nabla^2 p(\mathbb{1}) = \mathbb{1}d - 2\sum_{i=1}^n \nabla^2 p_i(\mathbb{1})$ (from Euler's identity: $d \cdot p(x) = \sum_{k=1}^n x_k \partial_k p(x)$). By induction and proposition 5.1, each matrix $\nabla^2 p_i(\mathbb{1})$ has at most one positive eigenvalue.

We work with the normalized Hessian matrix $\tilde{\nabla} p = \frac{1}{d-1} \operatorname{diag}(\nabla p(\mathbb{1}))^{-1} \nabla^2 p(\mathbb{1})$. Since the normalized Hessian matrix is stochastic, its top eigenvector is the all-ones vector. We change the inner product to work with the normalized Hessian now. For a d-homogeneous polynomial p with nonnegative coefficients and $d > 1$ and vectors $\phi, \psi \in \mathbb{R}^n$ define:

$$\langle \phi, \psi \rangle_p = (d-1) \sum_{j=1}^n \phi(j) \psi(j) (\partial_j p(\mathbb{1}))$$

giving us the norm $\|[\| \phi]_p^2 = \langle[\rangle \phi, \phi]_p$. It quite easily follows that:

$$\langle \phi, (\tilde{\nabla} p)\psi \rangle = \left\langle \phi, \nabla^2 p(\mathbb{1})\psi \right\rangle = \langle (\tilde{\nabla} p)\phi, \psi \rangle_p$$

In particular, $\tilde{\nabla}^2 p$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$ Moreover we have that:

$$\left\langle \phi, (\tilde{\nabla}^2 p)\psi \right\rangle_p = \left\langle \phi, \tilde{\nabla}^2 p(\mathbb{1})\psi \right\rangle = \frac{1}{d-2} \sum_{k=1}^{n} \left\langle \phi, \nabla^2 p_k(\mathbb{1})\psi \right\rangle = \frac{1}{d-2} \sum_{k=1}^{n} \left\langle \phi, \tilde{\nabla}^2 p_k \psi \right\rangle_{p_k}$$

<u>Note:</u> The Hessian $\nabla^2 p(\mathbb{1})$ can be viewed as the weighted adjacency matrix of a graph with edge weights $\partial_i \partial_j p(\mathbb{1})$. Thus, the normalized Hessian can be viewed as the associated random walk matrix and the inner product $\langle \cdot, \cdot \rangle$ can be viewed as a change of basis converting the random walk matrix into normalized adjacency matrix.

Take $\mu =$ eigenvalue of $\tilde{\nabla}^2 p$ with eigenvector ϕ. It is sufficient to show that $\mu \leq \mu^2$ for the inductive step since: First as $\tilde{\nabla}^2 p$ is stochastic we have that $\mu \leq 1$, which implies that $\mu = 1$ or $\mu \leq 0$. Thus to prove that $\tilde{\nabla}^2 p$ has exactly one positive eigenvalue, it is enough to show that $\lambda_2(\tilde{\nabla}^2 p) < 1$. But, since p is indecomposable, the underlying (weighted) graph of $\tilde{\nabla}^2 p$ is connected, so by 5.1, $\lambda(\tilde{\nabla}^2 p) < 1$.

So we are just left to prove that $\mu \leq \mu^2$. Note

$$\mu ||\phi||_p^2 = \left\langle \phi, (\tilde{\nabla}^2 p \phi) \right\rangle_p = \frac{1}{d-2} \sum_{k=1}^{n} \left\langle \phi, (\tilde{\nabla}^2 p_k)\phi \right\rangle_{p_k}$$

Decomposing ϕ orthogonally along $\mathbb{1}$ write it as:

$$\phi = \phi_k^{\perp \mathbb{1}} + \phi_k^{\mathbb{1}}$$

where $\phi_k^{\mathbb{1}} = \frac{\langle \phi, \mathbb{1} \rangle_{p_k}}{||\mathbb{1}||_{p_k}^2} \mathbb{1}$ and $\phi_k^{\perp \mathbb{1}}$ are orthogonal to $\mathbb{1}$. This implies that

$$\left\langle \phi_k^{\perp \mathbb{1}}, (\tilde{\nabla}^2 p_k)\phi_k^{\perp \mathbb{1}} \right\rangle_{p_k} \leq 0$$

since $\tilde{\nabla}^2 p_k$ has exactly one positive eigenvalue with corresponding eigenvector of $\mathbb{1}$. Therefore:

$$\mu ||\phi||_p^2 \leq \frac{1}{d-2} \sum_{k=1}^{n} \left\langle \phi_k^{\mathbb{1}}, (\tilde{\nabla}^2 p_k)\phi_k^{\mathbb{1}} \right\rangle_{p_k} = \frac{1}{d-2} \sum_{k=1}^{n} \frac{\langle \phi, \mathbb{1} \rangle_{p_k}^2}{\langle \mathbb{1}, \mathbb{1} \rangle_{p_k}} \tag{1}$$

Rewriting the numerator and denominator of each ratio in the right hand side, we have:

$$\langle \mathbb{1}, \mathbb{1} \rangle_{p_k} = (d-2) \sum_{i=1}^{n} (\partial_i p_k(\mathbb{1})) = (d-2)(d-1) \cdot p_k(\mathbb{1})$$

since $d \cdot p(x) = \sum_{k=1}^{n} x_k \partial_k p(x)$(Euler's identity). Moreover we get:

$$\langle \phi, \mathbb{1} \rangle_{p_k} = (d-2) \sum_{i=1}^{n} \phi(i)(\partial_i(\mathbb{1})) = (d-2) \cdot ((\nabla^2 p(\mathbb{1}))\phi)(k)$$

Thus putting the above identities together, we obtain:

$$\frac{\langle \phi, \mathbb{1} \rangle_{p_k}}{\langle \mathbb{1}, \mathbb{1} \rangle_{p_k}} = \frac{1}{(d-1) \cdot p_k} ((\nabla^2 p(\mathbb{1}))\phi)(k) = ((\tilde{\nabla}^2 p(\mathbb{1}))\phi)(k) = \mu \cdot \phi(k)$$

since ϕ is the eigenvector of $\tilde{\nabla}^2 p$ corresponding to μ. Plugging in the values in equation 1 we get

$$\begin{aligned} \mu ||\phi||_p^2 \leq \frac{1}{d-2} \sum_{k=1}^{n} \frac{\langle \phi, \mathbb{1} \rangle_{p_k}^2}{\langle \mathbb{1}, \mathbb{1} \rangle_{p_k}} &= \frac{\mu}{d-2} \sum_{k=1}^{n} \phi(k) \langle \phi, \mathbb{1} \rangle_{p_k} \\ &= \mu \sum_{k=1}^{n} \phi(k)((\nabla^2 p(\mathbb{1})\phi)(k)) = \mu \left\langle \phi, (\nabla^2 p(\mathbb{1}))\phi \right\rangle = \mu \left\langle \phi, (\tilde{\nabla}^2 p)\phi \right\rangle_p = \mu^2 ||\phi||_p^2 \end{aligned}$$

This implies that $\mu \leq \mu^2$ thus completing the induction and thereby proving the theorem. ■

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